EXPONENTIAL FORMULAE FOR SEMI-GROUP OF OPERATORS IN TERMS OF THE RESOLVENT

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ABSTRACT

Rate of convergence in terms of the modulus of continuity of either $T(t)f$ or of $T(t)Af$, where $T(t)$ is a strongly continuous semi-group of operators, is obtained for Phillips' and for Widder's exponential formula.

1. Introduction

In this paper $T(t)$ will be a strongly continuous semi-group of operators from a Banach space $\mathscr{B}$ into itself that is; a $\mathscr{C}_0$ semi-group; A will be its infinitesimal generator; and $R(\lambda; A) = (\lambda I - A)^{-1}$ the resolvent of A. Recalling that

$$(1.1) \qquad R(\lambda, A)f = \int_0^\infty e^{-\lambda t} T(t) f \, dt \qquad \lambda > \gamma_0$$

[6, p. 352], an inversion formula for the Laplace transform will express $T(t)f$ in terms of $R(\lambda; A)f$, in other words, will yield exponential formulae for $T(t)$.

It was shown by Phillips [8] that

$$(1.2) \qquad \lim_{\lambda \to \infty} e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda^2 t)^k}{k!} (R(\lambda; A))^k f \to T(t)f$$

for all $f \in \mathscr{B}$ uniformly in $t \in [0, L]$. Also, following Widder's inversion formula [9, Ch. VII], it was proved that for $f \in \mathscr{B}$ (see [6, p. 352])

$$(1.3) \qquad \lim_{k \to \infty} \left(\frac{k}{t} R\left(\frac{k}{t}; A \right) \right)^k f \to T(t) f$$

uniformly in $[0, L]$.

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We shall be interested in the rate of convergence of (1.2) and (1.3) in terms of the modulus of continuity in $[0, L]$, $w_L(\delta; T(\cdot)f)$ defined by

$$(1.4) \qquad w_L(\delta; T(\cdot)) = \sup\{\| T(t_1)f - T(t_2)f \|; |t_1 - t_2| \leqq \delta, 0 \leqq t_i \leqq L\}.$$

The rate of convergence of Hille's first and second exponential formulae [6, p. 354] in terms of (1.4) were investigated by Hsu [7], Butzer and Berens [1] and the author [2], [3]. The rate of convergence obtained in this paper is $Kw_L(\lambda^{-\frac{1}{2}}; T(\cdot)f)$ and $K_1 w_L(k^{-\frac{1}{2}}; T(\cdot)f)$ for some constants K and K_1 for (1.2) and (1.3) respectively. It will be shown that considering only $w_L(\delta; T(\cdot)f)$ the estimates for Widder's and Phillip's exponential formulae are up to a constant best possible, since $k^{-\frac{1}{2}}$ (or $\lambda^{-\frac{1}{2}}$) cannot be replaced by a function of k (or λ) tending faster to zero. However, when $d/dt(T(t)f)$ is continuous (or $f \in \mathscr{D}(A)$), a better estimate is possible in terms of the modulus of continuity of $d/dt(T(t)f = T(t)Af$; or modulus of continuity of $T(t)Af$. The rate in this case will be $K\lambda^{-\frac{1}{2}}w_L(\lambda^{-\frac{1}{2}}, T(\cdot)Af)$ (for some constant K) for (1.2) (and k replaces λ for (1.3)). We shall show that even if $d^m/dt^m(T(t)f)$ is continuous for all m (that is, $f \in \bigcap_m \mathscr{D}(A^m)$) we cannot improve on the results obtained by considering only $f \in \mathscr{D}(A)$.

2. Preliminary Lemmas

In this section we shall prove some Lemmas which will be essential for the proof of our rate of convergence estimates.

LEMMA 2.1. *Let $K(t,\tau;\lambda)$ be the positive measure defined by*

$$(2.1) \qquad K(t,\tau;\lambda) = e^{-\lambda(t+\tau)}\left\{\sum_{n=1}^{\infty} \frac{(\lambda^2 t)^n \tau^{n-1}}{(n!)^2} + \delta(\tau)\right\},$$

then for $\lambda > \alpha \geqq 0$

$$(2.2) \quad \int_0^\infty K(t,\tau;\lambda)(t-\tau)^2 e^{\alpha\tau} d\tau = e^{-\lambda t}\left\{\frac{2t}{\lambda-\alpha} - \frac{2}{(\lambda-\alpha)^2}\right\}$$

$$+ \left\{t^2\left[\left(\frac{\lambda}{\lambda-\alpha}\right)^2 - 1\right]^2 + \frac{2t}{\lambda-\alpha} + \frac{2}{(\lambda-\alpha)^2}\right\}\exp\left[\alpha t + (\alpha^2 t/\lambda - \alpha)\right]$$

and

$$(2.3) \qquad \int_0^\infty K(t,\tau;\lambda)(t-\tau)d\tau = \frac{1}{\lambda} + \frac{e^{-\lambda t}}{\lambda}.$$

PROOF. We shall calculate the integrals $\int_0^\infty K(t,\tau;\lambda)\tau^i e^{\alpha\tau}d\tau$ for $i = 0,1,2$ and both (2.1) and (2.2) will follow combining them. Fubini's theorem, applicable since $K(t,\tau;\lambda) - e^{-\lambda(t+\tau)}\,\delta(\tau) \geqq 0$, implies for $\lambda > \alpha \geqq 0$

$$(2.4)\quad \int_0^\infty K(t,\tau;\lambda)e^{\alpha\tau}d\tau = \sum_{n=1}^{\infty}\frac{\lambda^{2n}t^n e^{-\lambda t}}{(n!)^2}\int_0^\infty e^{-(\lambda-\alpha)\tau}\tau^n d\tau + e^{-\lambda t}$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\lambda^2 t}{\lambda-\alpha}\right)^n e^{-\lambda t} = \exp\left[\lambda t\left(-1+\frac{\lambda}{\lambda-\alpha}\right)\right] = \exp\left[\alpha t\left(1+\frac{\alpha}{\lambda-\alpha}\right)\right];$$

$$(2.5)\quad \int_0^\infty K(t,\tau;\lambda)\tau e^{\alpha\tau}d\tau = \sum_{n=1}^{\infty}\frac{\lambda^{2n}t^n e^{-\lambda t}(n+1)}{n!(\lambda-\alpha)^{n+1}}$$

$$= \left[\left(\frac{\lambda}{\lambda-\alpha}\right)^2 t + \frac{1}{\lambda-\alpha}\right]\exp\left(\alpha t\left(1+\frac{\alpha}{\lambda-\alpha}\right)\right) - \frac{e^{-\lambda t}}{\lambda-\alpha}$$

using $\tau\delta(\tau) = 0$; and

$$(2.6)\quad \int_0^\infty K(t,\tau;\lambda)\tau^2 e^{\alpha\tau}d\tau$$

$$= \left\{\frac{\lambda^4}{(\lambda-\alpha)^4}t^2 + \frac{1}{\lambda-\alpha}\left(\frac{\lambda}{\lambda-\alpha}\right)^2 4t + \frac{2}{(\lambda-\alpha)^2}\right\}\exp\left(\alpha t\left(1+\frac{\alpha}{\lambda-\alpha}\right)\right) - \frac{2e^{-\lambda t}}{(\lambda-\alpha)^2}.$$

Q.E.D.

In the particularly useful case $\alpha = 0$ we have as a corollary of Lemma 2.1

$$(2.7)\qquad \int_0^\infty K(t,\tau;\lambda)(t-\tau)^2 d\tau = \frac{2t}{\lambda} + \frac{2}{\lambda^2} + e^{-\lambda t}\left(\frac{2t}{\lambda} - \frac{2}{\lambda^2}\right).$$

Also, if λ is large enough we have

COROLLARY 2.1.a. *For $\lambda \geqq \lambda_0(\alpha;L)$ and $t \leqq L < \infty$*

$$(2.8)\qquad \int_0^\infty K(t,\tau;\lambda)(t-\tau)^2 e^{\alpha\tau}d\tau \leqq M\frac{1}{\lambda}$$

where M depends on λ_0, α and L.

Our next Lemma will be related to Widder's exponential formula (1.3).

LEMMA 2.2. *Let $W(k;t,\tau)$ be given by*

$$(2.9)\qquad W(k;t,\tau) \equiv \frac{1}{k!}\left(\frac{k}{t}\right)^{k+1} e^{-k\tau/t}\tau^k,$$

then for $k > \alpha t$

(2.10) $$\int_0^\infty W(k;t,\tau)e^{\alpha\tau}(t-\tau)^2 d\tau$$

$$= t^2\left(1 - \frac{2k+2}{k-\alpha t} + \frac{(k+1)(k+2)}{(k-\alpha t)^2}\right)\left(1-\frac{\alpha t}{k}\right)^{-k-1},$$

and

(2.11) $$\int_0^\infty W(k;t,\tau)(t-\tau)d\tau = -\frac{t}{k}.$$

PROOF. Following Lemma 2.1, we shall calculate $\int_0^\infty W(k;t,\tau)\tau^i e^{\alpha\tau}d\tau$ for $i = 0,1,2$ as follows:

(2.12) $$\int_0^\infty W(k;t,\tau)e^{\alpha\tau}d\tau = \frac{1}{k!}\left(\frac{k}{t}\right)^{k+1}\int_0^\infty e^{-(k\tau/t)+\alpha\tau}\tau^k d\tau$$

$$= \left(\frac{k}{t}\right)^{k+1}\left(\frac{k}{t}-\alpha\right)^{-k-1} = \left(1-\frac{\alpha t}{k}\right)^{-k-1};$$

(2.13) $$\int_0^\infty W(k;t,\tau)\tau e^{\alpha\tau}d\tau = t\left(\frac{k+1}{k-\alpha t}\right)\left(1-\frac{\alpha t}{k}\right)^{-k-1};$$

and

(2.14) $$\int_0^\infty W(k;t,\tau)\tau^2 e^{\alpha\tau}d\tau = t^2\frac{(k+1)(k+2)}{(k-\alpha t)^2}\left(1-\frac{\alpha t}{k}\right)^{-k-1}.$$

Combining (2.12), (2.13) and (2.14), we obtain (2.7) and (2.8).

The particular case of (2.10) when $\alpha = 0$ will be of use

(2.15) $$\int_0^\infty W(k;t,\tau)(t-\tau)^2 d\tau = \left(\frac{1}{k}+\frac{2}{k^2}\right)t^2.$$

The following simple corollary of (2.10) will be sufficient for most of our estimates.

COROLLARY 2.2.a. *Let $w(k;t,\tau)$ be given by (2.9), then for $k \geqq k_0(\alpha, L)$, $k_0 > \alpha L + 1$, and $t \leqq L$, we have*

(2.16) $$\int_0^\infty W(k;t,\tau)(t-\tau)^2 e^{\alpha\tau}d\tau \leqq M\frac{1}{k},$$

where M depends on α, L and k_0.

PROOF. The expression $(k/k-\alpha t)$ is bounded for $k \geqq \alpha t + 1$; $(1-\alpha t/k)^{-k-1}$ is bounded; and so is a polynomial in t where $t \in [0, L]$; which together imply that (2.15) follows from (2.12).

3. Rate of convergence depending on $w_L(\delta, T(\cdot)f)$.

We shall investigate first the rate of convergence of Phillips' exponential formula.

THEOREM 3.1. *Let $T(t)$ be a $\mathscr{C}_0$ semi-group, then for $0 \leqq t \leqq L - \delta$ (δ fixed) and $\lambda \geqq \lambda_0$ we have*

$$(3.1)\quad \left\| e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda^2 t)^n}{n!} (R(\lambda; A))^n f - T(t)f \right\| \leqq M(f, L) w_L(\lambda^{-\frac{1}{2}}, T(\cdot)f).$$

PROOF. Using (1.1) and recalling that for $n \geqq 1$, $\lambda \geqq \lambda_0$ and $\| T(t) \| \leqq K e^{\lambda_0 t}$ (which is always satisfied for some λ_0) we have

$$(3.2)\qquad (R(\lambda; A))^n = \frac{1}{n!} \int_0^\infty e^{-\lambda\tau} \tau^{n-1} T(\tau) d\tau \quad \text{for} \quad n \geqq 1,$$

while $(R(\lambda; A))^0 = I$. Therefore, we can write for $\lambda > \lambda_0$

$$(3.3)\quad I(\lambda, t, f) \equiv e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda^2 t)^n}{n!} (R(\lambda; A))^n f - T(t)f$$

$$= e^{-\lambda t} \int_0^\infty \sum_{n=1}^{\infty} \frac{(\lambda^2 t)^n}{(n!)^2} \tau^{n-1} e^{-\lambda\tau} [T(\tau)f - T(t)f] d\tau + e^{-\lambda t} [f - T(t)f],$$

where the change of order of integration and summation is justified for $\lambda > \lambda_0$ by their absolute convergence, and the insertion of $T(t)f$ under integral and sum signs is justified by (2.4) (substituting there $\alpha = 0$). For $0 \leqq t \leqq L - \delta$ and $\eta < \delta$ we can write

$$(3.4)\quad \| T(\tau)f - T(t)f \| \leqq w_L(|\tau - t|, T(\cdot)f) \leqq (1 + \eta^{-1} |t - \tau|) w_L(\eta, T(\cdot)f),$$

and, therefore, estimate $I(\lambda, t; f)$ as follows:

$$\| I(\lambda, t, f) \| \leqq \left\{ e^{-\lambda t} \int_0^L \sum_{n=1}^{\infty} \frac{(\lambda^2 t)^n}{(n!)^2} \left(1 + \frac{|\tau - t|}{\eta}\right) \tau^{n-1} e^{-\lambda\tau} d\tau \right.$$

$$\left. + e^{-\lambda t} \left(1 + \frac{|t|}{\eta}\right) \right\} w_L(\eta, T(\cdot)f) + e^{-\lambda t} \int_L^\infty \sum_{n=1}^{\infty} \frac{(\lambda^2 t)^n}{(n!)^2} (\| T(\tau)f \|$$

$$+ \| T(t)f \|) \tau^{n-1} e^{-\lambda\tau} d\tau.$$

Using the estimate $\| T(\tau)f \| \leqq K \| f \| e^{\lambda_1 \tau}$, the definition and positivity of $K(t, \tau; \lambda)$, and the monotonicity of $\delta^{-2}(t - \tau)^2$ in $L \leqq \tau < \infty$, we have

$$\| I(\lambda, t, f) \| \leqq \left\{ \int_0^\infty K(t, \tau; \lambda) d\tau + \eta^{-1} \int_0^\infty K(t, \tau; \lambda) |t - \tau| d\tau \right\} w_L(\eta, T(\cdot)f)$$

$$+ \frac{1}{\delta^2} \int_0^\infty K(t, \tau; \lambda)(t - \tau)^2 K \| f \| (e^{\lambda_0 \tau} + e^{\lambda_0 t}) d\tau = I_1 + I_2.$$

Since $\int_0^\infty K(t,\tau;\lambda)d\tau = 1$, we have, using Cauchy-Schwartz inequality and (2.8),

$$(3.5)\qquad \int_0^\infty K(t,\tau;\lambda)\left|t-\tau\right|d\tau \leqq \left\{\int_0^\infty K(t,\tau;\lambda)(t-\tau)^2 d\tau\right\}^{\frac{1}{2}} \leqq M^{\frac{1}{2}}\lambda^{-\frac{1}{2}}.$$

Choosing now $\eta = \lambda^{-\frac{1}{2}}$, we obtain

$$I_1 \leqq (1 + \lambda^{\frac{1}{2}}M^{\frac{1}{2}}\lambda^{-\frac{1}{2}})w_L(\lambda^{-\frac{1}{2}}, T(\cdot)f).$$

Corollary 2.1.a and simple properties of the modulus of continuity imply

$$I_2 \leqq M_1\frac{1}{\lambda} \leqq M_2 w_L\left(\frac{1}{\lambda}T(\cdot)f\right) \leqq M_2 w_L(\lambda^{-\frac{1}{2}}, T(\cdot)f),\ f \neq 0.$$

$$I_2 = 0 \text{ for } f = 0.$$

Combining the estimates for I_1 and I_2, we complete the proof of our theorem.

Our next result will be about the rate of convergence of Widder's Exponential formula.

THEOREM 3.2. *Let $T(t)$ be a $\mathscr{C}_0$ semi-group, then for $k \geqq k_0$ and, $0 < t \leqq L-\delta$, we have*

$$(3.6)\qquad \left\|\left(\frac{k}{t}R\left(\frac{k}{t}, A\right)\right)^{k+1}f - T(t)f\right\| \leqq M(\delta, L)w_L(k^{-\frac{1}{2}}, T(\cdot)f).$$

PROOF. Following (3.2) and $\|T(t)\| \leqq ke^{\lambda_0 t}$, we have

$$(3.7)\qquad \left(\frac{k}{t}R\left(\frac{k}{t}, A\right)\right)^{k+1} = \left(\frac{k}{t}\right)^{k+1}\cdot\frac{1}{k!}\int_0^\infty e^{-k\tau}\tau^k T(\tau)d\tau,$$

where the integral converges for $k/t > \lambda_0$ or $k > \lambda_0 t$. Choosing $k_0 > \lambda_0 L + 1$, we can proceed as in the proof of Theorem 3.1., using Lemma 2.2 instead of 2.1. In fact, wherever Eq. (2.l) ($l = 1,\cdots,8$) is used there, we should us here Eq. (2.$l+8$) and replace λ by k. Q.E.D.

4. Rate of convergence for $f \in \mathscr{D}(A)$.

In this section the rates of convergence of the exponential formulae (1.2) and (1.3) are related to the modulus of continuity of $(d/dt)T(t)f$. We would not treat connections with the modulus of continuity of higher derivatives, since, as we shall see later, for the formulae treated, the estimates would not improve by more then a multiplicative constant.

THEOREM 4.1. *Let $T(t)$ be a $\mathscr{C}_0$ semi-group, then if $(d/dt)T(t)f = T(t)Af$ is continuous, we have for $0 < t \leqq L-\delta$*

$$(4.1)\qquad \left\| e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda^2 t)^n}{n!}(R(\lambda;A))^n f - T(t)f \right\| \leqq M_1 \lambda^{-\frac{1}{2}} w_L(\lambda^{-\frac{1}{2}}, T(\cdot)Af) \quad \lambda > \lambda_0$$

and

$$(4.2)\qquad \left\| \left(\frac{k}{t} R\left(\frac{k}{t}; A\right)\right)^{k+1} f - T(t)f \right\| \leqq M_2 k^{-\frac{1}{2}} w_L(k^{-\frac{1}{2}}, T(\cdot)Af) \quad k > k_0.$$

PROOF. If $T(t)Af$ is continuous, we write

$$T(\tau)f - T(t)f = (\tau - t)\int_0^1 T(t + \zeta(\tau - t))Af\,d\zeta$$

$$= (\tau - t)T(t)Af + (\tau - t)\int_0^1 [T(t + \zeta(\tau - t))Af - T(t)Af]d\zeta.$$

We shall prove (4.1) first. Following (3.3) in the proof of Theorem 3.1 we have for $t \leqq L - \delta$

$$I(\lambda, t, f) = T(t)Af \int_0^\infty K(t, \tau; \lambda)(\tau - t)d\tau$$

$$+ \int_0^\infty K(t, \tau; \lambda)(\lambda - t) \left\{ \int_0^1 [T(t + \zeta(\tau - t))Af - T(t)Af]d\zeta \right\} d\tau \equiv I_1 + I_2.$$

Using 2.3, we estimate I_1 by

$$(4.3)\qquad \| I_1 | \leqq \| T(t) \| \| Af \| \cdot \left(\frac{1}{\lambda} + \frac{e^{-\lambda t}}{\lambda}\right) \leqq 2 \| T(t) \| \; \| Af \| \cdot \frac{1}{\lambda}.$$

To estimate I_2 we write

$$\| I_2 \| \leqq \int_0^L K(t, \tau; \lambda) | \tau - t | \int_0^1 \| T(t + \zeta(\tau - t))Af - T(t)Af \| d\zeta\} d\tau$$

$$+ \int_L^\infty K(t, \tau; \lambda) | \tau - t | \left\{ \| T(t)Af \| + \int_0^1 \| T(t + \zeta(\tau - t))Af \| d\zeta \right\} d\tau$$

$$\equiv J_1 + J_2.$$

To estimate J_1 we estimate first the inner integral

$$\int_0^1 | T(t + \zeta(\tau - t))\, Af - T(t)Af \| d\zeta \leqq \int_0^1 w_L(\zeta | \tau - t |, T(\cdot)Af)d\zeta$$

$$\leqq \int_0^1 w_L(| \tau - t |; T(\cdot)Af)d\zeta = w_L(| \tau - t |; T(\cdot)Af)$$

$$\leqq (1 + \eta^{-1} | \tau - t |) w_L(\eta; T(\cdot)Af)).$$

Therefore,

$$J_1 \leqq \int_0^L K(t,\tau;\lambda)\left|\tau - t\right|(1+\eta^{-1}\left|\tau - t\right|)w_L(\eta;T(\cdot)Af)d\tau$$

$$\leqq w_L(\eta;T(\cdot)Af)\left\{\int_0^\infty K(t,\tau;\lambda)\left|t-\tau\right|d\tau + \eta^{-1}\right.$$

$$\left.\times\int_0^\infty K(t,\tau;\lambda)(t-\tau)^2 d\tau\right\}.$$

Choosing $\eta = \lambda^{-\frac{1}{2}}$ and recalling (2.8) and (3.5), we obtain

$$\text{(4.4)}\quad J_1 \leqq \left(M^{\frac{1}{2}}\lambda^{-\frac{1}{2}} + \lambda^{\frac{1}{2}}M\frac{1}{\lambda}\right)w_L(\lambda^{-\frac{1}{2}};T(\cdot)Af) \leqq K\lambda^{-\frac{1}{2}}w_L(\lambda^{-\frac{1}{2}};T(\cdot)Af).$$

To estimate J_2 we recall that $\| T(\tau)Af \| \leqq \| T(\tau)\| \, \| Af\| \leqq M\| Af\| \, e^{\lambda_0 t}$ and therefore, we have, using the monotonicity of $\left|\tau - t\right|$ for $\tau \in [L,\infty)$ and the positivity of $K(t,\tau,\lambda)$,

$$J_2 \leqq \int_L^\infty K(t,\tau;\lambda)\left|\tau - t\right| M \left\| Af \right\| (e^{\lambda_0 t} + e^{\lambda_0(t+\tau)})d\tau$$

$$\leqq M\left\| Af \right\| e^{\lambda_0 t}\frac{1}{\delta}\int_L^\infty K(t,\tau;\lambda)(\tau - t)^2(1+e^{\lambda_0\tau})d\tau$$

$$\leqq M\left\| Af \right| e^{\lambda_0 L}\frac{1}{\delta}\int_0^\infty K(t,\tau;\lambda)(\tau - t)^2(1+e^{\lambda_0\tau})d\tau \leqq K_2\cdot\lambda^{-1}$$

$$\leqq K_2\lambda^{-\frac{1}{2}}w_{1,L}(\lambda^{-\frac{1}{2}},T(\cdot)f).$$

Combining the above estimates, we complete the proof of (4.1). The proof of (4.2) is analogous using $(2, l+8)$ wherever $(2, l)$ is used in the above.

5. On the generality of estimates in Sections 3 and 4.

We shall first show that the estimate in Theorem 3.2 is the best possible up to a multiplicative constant. To demonstrate the above, let us consider the example of the Banach space $C_0(-\infty,\infty)$ of bounded continuous functions, which tend to zero as x tends to infinity, and the semi-group $T(t)f(x) = f(t+x)$. We choose $f_1(x) = \begin{cases} 1-\left|x\right| & \left|x\right| \leqq 1 \\ 0 & \text{otherwise} \end{cases}$, and for this choice obviously $w_L(\delta, T(\cdot)f) = \delta$ (valid for any L). To compute a lower estimate for (3.6) on the other hand we estimate

$$(5.1)\qquad \sup_x \left|\left(\frac{k}{t}\right)^{k+1}\frac{1}{k!}\int_0^\infty e^{-k\tau/t}\,{}^k f_1(x+\tau)d\tau - f_1(x+t)\right|.$$

We justify the statement above if we show for some x the expression in (5.1) is greater then $Kk^{-\frac{1}{2}}$, where $K > 0$ is independent of k. Choosing $x = -1$, $t = 1$, and $0 < \delta < 1$, we have

$$(k)^{k+1}\frac{1}{k!}\int_0^\infty e^{-k\tau}\tau^k[f_1(\tau-1)-f_1(0)]d\tau = I_1 + I_2$$

where I_1 and I_2 are integrals on $|\tau - 1| \leqq \delta$, and $|\tau - 1| > \delta$ respectively. For $|\tau| \leqq \delta \quad f_1(\tau-1)-f_1(0) = -|\tau-1|$, and, therefore,

$$|I_1| = k^{k+1}\frac{1}{k!}\int_{|1-\tau|\leqq\delta} e^{-k\tau}\tau^k|\tau-1|d\tau.$$

Using Theorem 4.3 of [5] and especially (13.3) there, we obtain for $a_k = 0$, $g(k) = k$, $\lambda = 0$, and $h(x)$ being $h(\tau)$ here and $h(\tau) = -\tau + \log\tau$, $a = 1$, $b = 1+\delta$, the estimate

$$k^{\frac{1}{2}}\cdot k^{k+1}\frac{1}{k!}\int_1^{1+\delta} e^{-k\tau}\tau^k(\tau-1)d\tau \sim A > 0 \quad k\to\infty.$$

A similar estimate is valid for $\int_{1-\delta}^1$ and, therefore, $|I_1| \geqq K'k^{-\frac{1}{2}}$ for $k \geqq k_0$. I_2 can be estimated by (2.15)

$$|I_2| \leqq \int_{\substack{|1-\tau|\geqq\delta\\ \tau\geqq 0}} W(k;1,\tau)d\tau \leqq \frac{1}{\delta^2}\int_0 (1-\tau)^2W(k;1,\tau)d\tau \leqq \frac{1}{\delta^2}\left(\frac{1}{k}+\frac{2}{k^2}\right)$$

But

$$\left|k^{k+1}\frac{1}{k!}\int_0^\infty e^{-k\tau}\tau^k[f_1(\tau-1)-f_1(0)]d\tau\right| \geqq |I_1| - |I_2|,$$

which completes the proof that the expression in (5.1) is greater than $Kk^{-\frac{1}{2}}$.

To show that the result of Theorem 3.1 is also best possible we use the same space, semi-group and function as in the above. We have to show that

$$(5.2)\qquad \int_{|\tau-1|\leqq\delta} K(1,\tau,\lambda)|\tau-1|d\tau \geqq c\lambda^{-\frac{1}{2}} \text{ for some } c>0\ \lambda\geqq\lambda_0.$$

Following calculations in Lemma 2.1 we obtain after messy derivations of $\int_0^\infty K(t,\tau,\lambda)\tau^i d\tau \quad i = 0,1,2,3,4$

$$(5.3)\ \int_0^\infty K(t,\tau,\lambda)(t-\tau)^4d\tau = \frac{12t^2}{\lambda^2}+\frac{72t}{\lambda^3}+\frac{24}{\lambda^4}+e^{-\lambda t}\left(\frac{4t^3}{\lambda}-\frac{12t^2}{\lambda^2}+\frac{24t}{\lambda^3}-\frac{24}{\lambda^4}\right).$$

In particular $\int_0^\infty K(1,\tau,\lambda)(1-\tau)^4 d\tau = (12/\lambda^2) + o(1/\lambda^2)$.

Using (2.7) and Cauchy-Schwartz's inequality we have

$$\begin{aligned}\frac{2}{\lambda} + o\left(\frac{1}{\lambda}\right) &= \int_0^\infty K(1,\tau,\lambda)(1-\tau)^2 d\tau \\ &\leqq \left\{\int_0^\infty K(1,\tau,\lambda)\left|1-\tau\right|^3 d\tau\right\}^{\frac{1}{2}}\left\{\int_0^\infty K(1,\tau,\lambda)\left|1-\tau\right| d\tau\right\}^{\frac{1}{2}} \\ &\leqq \left\{\int_0^\infty K(1,\tau,\lambda)\left|1-\tau\right|^4 d\tau\right\}^{\frac{1}{4}}\left\{\int_0^\infty K(1,\tau,\lambda)\left|1-\tau\right|^2 d\tau\right\}^{\frac{1}{4}} \\ &\qquad \times\left\{\int_0^\infty K(1,\tau,\lambda)\left|1-\tau\right| d\tau\right\}^{\frac{1}{2}}\end{aligned}$$

which implies

$$\begin{aligned}\left\{\int_0^\infty K(1,\tau,\lambda)\left|1-\tau\right| d\tau\right\}^{\frac{1}{2}} &\geqq \left(\frac{2}{\lambda}+o\left(\frac{1}{\lambda}\right)\right)\left(\frac{12}{\lambda^2}+o\left(\frac{1}{\lambda^2}\right)\right)^{-\frac{1}{4}}\cdot\left(\frac{2}{\lambda}+o\left(\frac{1}{\lambda}\right)\right)^{-\frac{1}{4}} \\ &= \frac{2}{\sqrt[4]{24}}\frac{1}{\lambda^{\frac{1}{4}}} + o\left(\frac{1}{\lambda^{\frac{1}{4}}}\right) \qquad \lambda\to\infty\end{aligned}$$

or
$$\int_0^\infty K(1,\tau,\lambda)\left|1-\tau\right| d\tau \geqq \frac{2}{\sqrt{6}}\frac{1}{\lambda^{\frac{1}{2}}} + o\left(\frac{1}{\lambda^{\frac{1}{2}}}\right) \qquad \lambda\to\infty.$$

Since

$$\begin{aligned}\int_{|1-\tau|\geqq\delta} K(1,\tau,\lambda)\left|1-\tau\right| d\tau &\leqq \frac{1}{\delta}\int_{|1-\tau|\geqq\delta} K(1,\tau,\lambda)(1-\tau)^2 d\tau \\ &\leqq \frac{1}{\delta}\int_0^\infty K(1,\tau,\lambda)(1-\tau)^2 d\tau \\ &= \frac{1}{\delta}\left[\frac{2}{\lambda}+o\left(\frac{1}{\lambda}\right)\right] \qquad \lambda\to\infty\end{aligned}$$

we can easily conclude the proof of (5.2).

We shall show now that if in Theorem 4.1 we assumed the existence and continuity of all derivatives of $T(t)f$, we would not improve our estimate.

Let us treat the Banach space of continuous functions in $[0,\infty)$, satisfying $\lim_{x\to\infty} e^{-x}f(x) = 0$, with the norm $\|f\| = \sup\left|e^{-x}f(x)\right|$, and let $T(t)f(x) = f(x+t)$. Choosing now $f_1(x) = x^2$, obviously $T(t)f_1$ has all derivatives. Using the lemmas of Section 2 and Theorem 4.1, we get

$$I(\lambda) = \left\| e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda^2 t)^n}{n!} R(\lambda; A)^n f_1 - T(t) f_1 \right\|$$

$$= \sup_x \left| e^{-\lambda t} \int_0^\infty \sum_{n=1}^{\infty} \frac{(\lambda^2 t)^n}{(n!)^2} \tau^{n-1} [(\tau + x)^2 - (t + x)^2] \, d\tau \right.$$
$$\left. + e^{-\lambda t} [x^2 - (t + x)^2] \right|.$$

Let us choose $x = 1$, then we have by (2.4), (2.5) and (2.6)

$$|I(\lambda)| \geqq \int_0^\infty K(t, \tau, \lambda)[(\tau^2 - t^2) + 2(\tau - t)] d\tau$$
$$= \frac{4t}{\lambda} + \frac{2}{\lambda^2} - \frac{2e^{-t}}{\lambda^2} + 2\left[\frac{1}{\lambda} - \frac{e^{-\lambda t}}{\lambda}\right] \geqq \frac{4t}{\lambda}.$$

Similarly and using the same example, we show that (4.2) cannot be improved:

$$I(k) = \left\| \left(\frac{k}{t}\right) R\left(\frac{k}{t}; A\right)^{k+1} f_1 - T(t) f_1 \right\|$$
$$= \sup_x \left| \int_0^\infty W(k; t, \tau)((\tau + x)^2 - (t + x)^2 d\tau \right|.$$

Choosing $x = 1$ and using (2.12), (2.13) and (2.14), we obtain

$$I(k) = \int_0^\infty W(k; t, \tau)(\tau^2 - t^2) + 2(\tau - t)) d\tau$$
$$= t^2 \left(\frac{(k+1)(k+2)}{k^2} - 1\right) + 2t\left(\frac{k+1}{k} - 1\right) = t^2 \frac{3}{k} + \frac{2t}{k} + \frac{2t^2}{k^2} \geqq \frac{2t}{k}.$$

6. Remarks, generalizations and corollaries

In this section we make a few remarks that enable us to generalize somewhat theorems of former sections.

REMARK 6.1. The modulus of continuity $w_L(\delta, T(\cdot)f)$ depends on the modulus of continuity in $(0, \eta]$ $w_\eta(\delta, \tau(\cdot)f)$, $\eta > \delta$, since

(6.1) $$w_L(\delta; T(\cdot)f) \leqq \sup_{t \leqq L-\eta} \| T(t) \| \, w_\eta(\delta; T(\cdot)f) \leqq M e^{\alpha(L-\eta)} w_\eta(\delta; T(\cdot)f).$$

Therefore, in Theorems 3.1 and 3.2 the rate of convergence could be estimated by $K(\eta) w_\eta(\lambda^{-\frac{1}{2}}; T(\cdot)f)$ and $M(\eta) w_\eta(k^{-\frac{1}{2}};\ T(\cdot)f)$ respectively. However, since (6.1) in many cases is far from being best possible, we may have a worse estimate using it instead of $w_L(\delta; T(\cdot)f)$.

REMARK 6.2. Following remark 6.1, we can also estimate $w_L(\delta; T(\cdot)Af)$ by $w_\eta(\delta, T(\cdot)Af)$ $\eta > \delta$ following the above

$$(6.2) \qquad w_L(\delta; T(\cdot)Af) \leqq Me^{\alpha(L-\eta)}w_\eta(\delta; T(\cdot)Af,$$

and therefore can give the estimates of Theorem 4.1 in terms of

$$\lambda^{-\frac{1}{2}}w_\eta(\lambda^{-\frac{1}{2}}; T;\ T(\cdot)Af).$$

REMARK 6.3. It may be useful on some occasions to consider the modulus of continuity in $[a,b]$, instead of $(0,L)$, denoted by

$$(6.3) \quad w(\delta; T(\cdot)f,[a,b]) = \sup\{\|T(t_1)f - T(t_2)f\|; |t_1 - t_2| < \delta, a \leqq t_1, t_2 \leqq b\}.$$

We can observe easily a difference if we consider the semi-group of left translations on continuous functions on $[0,\infty)$, such that

$$\|f\| = \sup_{x \geqq 0} |e^{-x}f(x)|, \text{ and consider } f_1(x) = \begin{cases} 0 & 0 \leqq x \leqq 1 \\ (x-1)^\alpha & 1 < x < \infty \end{cases}$$

$w(\delta; T(\cdot)f_1) > \delta^\alpha$ while $w(\delta; T(\cdot)f_1, [2,3]) < K \cdot \delta$. With slight modifications in the proof we have, instead of Theorems 3.1 and 3.2, for $t \in [a+\eta, b-\eta]$

$$(6.5) \qquad \left\| e^{-\lambda t} \sum_{h=0}^{\infty} \frac{(\lambda^2 t)^n}{n!}(R(\lambda; A))^n f - T(t)f \right\| \leqq Mw(^{-\frac{1}{2}}, T(\cdot)f, [a,b]),$$

and

$$(6.6) \qquad \left\| \left(\frac{k}{t}R\left(\frac{k}{t}; A\right)\right)^{k+1} f - T(t)f \right\| \leqq Mw(k^{-\frac{1}{2}}, T(\cdot)f,[a,b]),$$

where M depends on f, on $[a,b]$, and on the estimate $\|T(t)\| \leqq Ke^{\alpha t}$.

REMARK 6.4. Similar modifications to those mentioned in Remark 6.3 about taking the modulus of continuity only on $[a,b]$, can be applied in Theorem 4.1 to the modulus of continuity of $(d/dt)T(t)f$. In fact, $T(t)f$ does not necessarily have a derivative in $(0,a]$ or $f \notin \mathscr{D}(A)$ but $T(a)f \in A$. We shall obtain for $t \in [a+\eta, b-\eta]$

$$(6.7) \qquad \left\| \left(\frac{k}{t}R\left(\frac{k}{t}; A\right)\right)^{k+1} f - T(t)f \right\| \leqq Mk^{-\frac{1}{2}}w_{b-a}(k^{-\frac{1}{2}}, \tau(\cdot)AT(a)f),$$

and corresponding formula for the Phillips exponential formula.

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